Name of College - S.S. College, J-Bad
Deptt - Mathematics
Topic - Cauchy's Mean Value theorem and problem based on it.
Class - B.Sc II
Time - 9AM to 10.30
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Cauchy's Mean Value Theorem

Statement :-> If two functions $f(x)$ and $g(x)$ Satisfy the Conditions Given below

(i) $f(x)$ and $g(x)$ are continuous in $[a,b]$

(ii) $f'(x)$ and $g'(x)$ Exists and are continuous in (a,b)

Also (iii) $g'(x) \neq 0$ for any point in (a,b), then there Exists at Least one point c in (a,b) for which

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof :-> Let an Auxiliary function $\psi(x)$ such that

$$\psi(x) = f(x) - \lambda g(x) \quad \text{——— (i)}$$

Now λ is so choosen such that

$$\psi(a) = \psi(b)$$

i.e. $f(a) - \lambda g(a) = f(b) - \lambda g(b)$

$\Rightarrow f(a) - f(b) = \lambda g(a) - \lambda g(b)$

$$\Rightarrow \frac{f(a)-f(b)}{g(a)-g(b)} = \lambda \quad \text{——— (ii)}$$

Since, we have $g'(x) \neq 0$ for any point in (a,b)

therefore $g(b) \neq g(a)$

as otherwise $g(b) = g(a)$,

the function $g(x)$ would satisfy all conditions of Rolle's theorem in (a,b) and as such $g'(x) = 0$ at some point in (a,b)

Which is against hypothesis i.e $g'(x) \neq 0$ for any point in (a,b)

thus from (ii) λ is always finite and determinate

The function $\varphi(x)$ thus satisfies all condition of Rolle's theorem and as $\varphi'(c)=0$ where $c\in(a,b)$

$$\Rightarrow f'(c)-\lambda g'(c)=0$$

$$\Rightarrow f'(c)=\lambda g'(c)$$

$$\Rightarrow \frac{f'(c)}{g'(c)}=\lambda=\frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{——— (iii)}$$

with the help of (ii) which proves the Cauchy's Mean value theorem

Cor $\rightarrow$ If $f(a)=g(a)=0$

$$\Rightarrow \frac{f'(c)}{g'(c)}=\frac{f(b)}{g(b)} \quad \text{where } c\in(a,b)$$

Remarks :$\rightarrow$ (i) $f'(x)$ and $g'(x)$ should not be equal to zero for some value of x

(ii) If $g(x)=x$ then Cauchy's form of the mean value theorem reduces to Lagrange's mean value theorem.

(iii) Cauchy's Mean value theorem can not be deduced from Lagrange's Form of Mean value theorem

Problem based on Cauchy's Mean value theorem

(1) Verify Cauchy's Mean value theorem for the function $f(x)=x^2$ and $g(x)=x$ in $[a,b]$

Solution :$\rightarrow$ Here $f(x)=x^2$
$g(x)=x$

Since here $f(x)$ and $g(x)$ are polynomial and therefore these are continuous and differentiable for all values of x i.e.

i.e $f(x)$ and $g(x)$ are continuous in $[a,b]$ and $f'(x)$ and $g'(x)$ exists in (a,b)

Since $g(x) = x \Rightarrow g'(x) = 1 \neq 0$ for any $x \in (a,b)$

and therefore $f(x)$ and $g(x)$ satisfy all the conditions of Cauchy's Mean value theorem. and therefore there exists a point $c \in (a,b)$ such that-

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$\because f(x) = x^2 \Rightarrow f(a) = a^2$, $f(b) = b^2$

$g(x) = x$, $g(a) = a$, $g(b) = b$

$f'(x) = 2x$ $\quad g'(x) = 1$

$f'(c) = 2c$ $\quad g'(c) = 1$

$$\therefore \frac{b^2 - a^2}{b-a} = \frac{2c}{1}$$

$\Rightarrow 2c = b+a \Rightarrow c = \frac{a+b}{2} \in (a,b)$

thus Cauchy Mean value theorem is verified

Ex:→ Find c of Cauchy's Mean value theorem for the function $f(x) = \sqrt{x}$ $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$

Solution. →

By the question

$f(x) = \sqrt{x}$ $g(x) = \frac{1}{2\sqrt{x}}$

Here we find

(i) $f(x)$ and $g(x)$ are continuous in the closed interval $[a, b]$

(ii) $f'(x)$ and $g'(x)$ exists in (a, b)

(iii) $g'(x) = -\frac{1}{2} x^{-3/2} \neq 0$ for any $x \in (a, b)$

$\therefore$ $f(x)$ and $g(x)$ ~~satisfies~~ satisfy all the required conditions of Cauchy's mean value theorem

Hence there exists a point $c \in (a, b)$

Such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$ —— (1)

Here $f(x) = \sqrt{x} = x^{1/2}$ $\quad g(x) = \frac{1}{2\sqrt{x}}$

$f(a) = a^{1/2}$ $\quad g(a) = \frac{1}{\sqrt{a}}$

$f(b) = b^{1/2}$ $\quad g(b) = \frac{1}{\sqrt{b}}$

$f(b) - f(a) = b^{1/2} - a^{1/2}$ $\quad g(b) - g(a) = \frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}$

$f'(c) = \frac{1}{2\sqrt{c}}$ $\quad g(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$g'(x) = -\frac{1}{2} x^{-3/2}$

$\therefore$ From (1)

$g'(c) = -\frac{1}{2} c^{-3/2}$

$$\frac{b^{1/2} - a^{1/2}}{-\frac{\sqrt{b} - \sqrt{a}}{\sqrt{ab}}} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2} c^{3/2}}$$

$= -\frac{1}{2c^{3/2}}$

$\sqrt{ab} = c^{3/2} \cdot c^{1/2} = c$

$\Rightarrow c = \sqrt{a} \cdot \sqrt{b}$